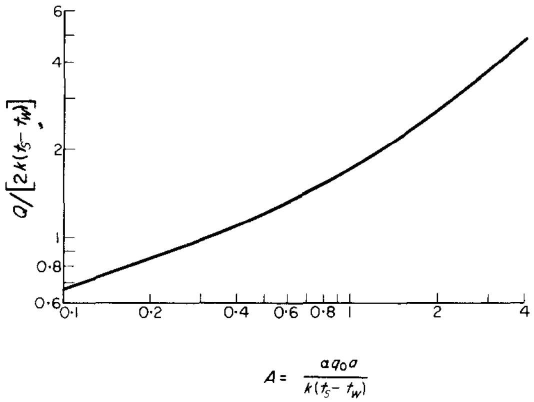

FIG. 8. Dimensionless heat flow through frozen region as a function of physical parameter involving absorbed incident radiation.

The results are given in Figs. 7 and 8 which show the configurations of the free boundary and the heat flow through the region as a function of the physical parameter A. The large values of A are associated with large values of the absorbed radiative heat flux, or small values of the cooling temperature below the surface temperature. These conditions yield thin conducting regions. The large A are also associated with large values of the dimensionless heat flow as shown in Fig. 8.

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APPLICATION OF A SAMPLED-DATA MODEL TO THE TRANSIENT RESPONSE OF A DISTRIBUTED PARAMETER SYSTEM SUBJECT TO SIMULTANEOUS RADIATION AND CONVECTION

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INTRODUCTION

A DISTRIBUTED parameter system represented by the one-dimensional heat conduction equation subject to both radiation and convection is studied. In the analysis the input function is assumed to be sampled and held and by employing the Laplace transform and the z-transforms respectively, a discrete-time system of equations is obtained for digital computer solution. Results obtained were in excellent agreement with published ones of Crosbie and Viskanta [1]. The method of solution is a direct one and no iteration techniques are required.

STATEMENT OF THE PROBLEM

We shall be concerned with the problem of obtaining the transient heating and cooling solutions for the one-dimensional slab initially at a uniform distribution and then subjected to both radiation and convection at one of its boundaries. The assumptions made and the nomenclature used are identical to those of Crosbie and Viskanta [1] and hence will not be repeated here. The basic system equation is then given by

$$\frac{\partial u}{\partial t}(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t); \qquad \begin{matrix} t > 0 \\ 0 < x < 1 \end{matrix} \tag{1}$$

and the initial and boundary conditions are

$$u(x, 0) = u_i \tag{2a}$$

$$\frac{\partial u}{\partial x}(0, t) = 0 \tag{2b}$$

$$\frac{\partial u}{\partial x}(1, t) = -\,g[u(1, t), t]. \tag{2c}$$

Here $u(x, t)$ is the temperature of the system as a function of the time t and the spatial coordinate x. The dimensionless temperatures and the heat flux are defined in Table 1 of [1].

Formation of model equations

Since equation (1) is linear, there results upon Laplace transforming same, the following ordinary differential equation

$$su(x,s) = \frac{d^2u}{dx^2}(x,s). \quad (3)$$

Here the initial condition is assumed zero; the initial condition as given by equation (2a) may be incorporated into the model equations. The solution to equation (3) is

$$u(x,s) = C_1 \cosh[(\sqrt{s})\,x] + C_2 \sinh[(\sqrt{s})\,x]. \quad (4)$$

Employing the boundary condition at $x = 0$ gives $C_2 = 0$. Assuming the input function at the boundary $x = 1$ is sampled and held yields the following

$$\frac{du}{dx}(1,s) = \frac{(1-e^{-sP})}{s}\sum_{k=0}^{\infty} -g[u(1,kP)]\,e^{-kPs} \quad (5)$$

Employing equation (5) to obtain C_1 yields the following solution

$$u(x,s) = (1-e^{-sP})\frac{\cosh[(\sqrt{s})x]}{s(\sqrt{s})\sinh[(\sqrt{s})]}\sum_{k=0}^{\infty} -g[u(1,kP)]\,e^{-kPs}. \quad (6)$$

The inverse of (6) is given by

$$u(x,kP) = (1-e^{-sP})\Bigg\{\sum_{k=0}^{\infty} kPg(kP)e^{-kPs} + \sum_{k=0}^{\infty} g(kP)\,H_0(x)\,e^{-kPs} + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} g(kP)\exp(-n^2\pi^2kP)\,H_n(x)\,e^{-kPs}\Bigg\} \quad (7)$$

where t has been replaced by the discrete-time kP, $-g[u(1,kP)]$ by $g(kP)$ and

$$H_0(x) = (3x^2-1)/6 \quad (8)$$

$$H_n(x) = \frac{-2}{\pi^2}\frac{(-1)^n}{n^2}\cos(n\pi x); \quad n = 1,2,\ldots,\infty. \quad (9)$$

To obtain equation (7) in discrete-time form the z transform is employed. Hence

$$z = e^{sP}. \quad (10)$$

Hence equation (7) becomes

$$u(x,z) = \frac{z-1}{z}\Bigg\{\sum_{k=0}^{\infty} g(kP)\,kP\,z^{-k} + \sum_{k=0}^{\infty} g(kP)\,H_0(x)\,z^{-k} + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} g(kP)\exp(-n^2\pi^2kP)\,H_n(x)\,z^{-k}\Bigg\}. \quad (11)$$

Employing the closed form for the infinite series in z, there results

$$u(x,z) = \frac{Pg(kP)}{z-1} + g(kP)\,H_0(x) + \sum_{n=1}^{\infty} g(kP)\,H_n(x)\frac{z-1}{z-h_n} \quad (12)$$

where

$$h_n = \exp(-n^2\pi^2P). \quad (13)$$

Next, the number of terms to retain in the infinite series in n is readily determined by observing the decay rate of the various exponential terms. If q terms are retained, then the coefficient $H_q(x)$ is set so as to satisfy the assumed zero initial condition. This will involve negligible error. Hence

$$H_q(x) = -[H_0(x) + H_1(x) + \ldots + H_{q-1}(x)]. \quad (14)$$

Also

$$H_n(x)\left[\frac{z-1}{z-h_n}\right] = H_n(x) + \frac{H_{nn}(x)}{z-h_n} \quad (15)$$

where

$$H_{nn}(x) = -H_n(x)\,[1-h_n]. \quad (16)$$

Hence employing equations (14)–(16), equation (12) may be written as the following upon multiplying numerator and denominator by z^{-1}

$$u(x,z) = Pz^{-1}\frac{g(kP)}{1-z^{-1}} + \sum_{j=1}^{q} H_{jj}(x)\,z^{-1}\frac{g(kP)}{1-z^{-1}h_j}. \quad (17)$$

Let

$$W_j(z) = g(kP)/(1-z^{-1}h_j); \quad j = 0,1,2,\ldots,q. \quad (18)$$

Then

$$W_j(z) = g(kP) + z^{-1}h_j\,W_j(z). \quad (19)$$

Referring to Fig. 1 the state equations are then given by (here the output of each z^{-1} block is by definition a state variable)

$$\begin{aligned} W_0(k+1)P &= W_0(kP) + g(kP)\\ W_1(k+1)P &= h_1\,W_1(kP) + g(kP)\\ &\;\;\vdots\\ W_q(k+1)P &= h_q\,W_q(kP) + g(kP)\end{aligned} \quad (20)$$

with the temperature $u(x,kP)$ at any location x given by

$$u(x,kP) = P\,W_0(kP) + H_{11}(x)\,W_1(kP) + \ldots + H_{qq}(x)\,W_q(kP). \quad (21)$$

Hence for any value of x desired the discrete-time equations may be obtained. Actually one may proceed directly from the characteristic solution as given by equation (7) and write the state equations once a value of the sampling period P has been chosen and also the number of terms q in the infinite series, as the coefficients h_j follow a regular pattern and the coefficient associated with the input $g(kP)$ is always unity.

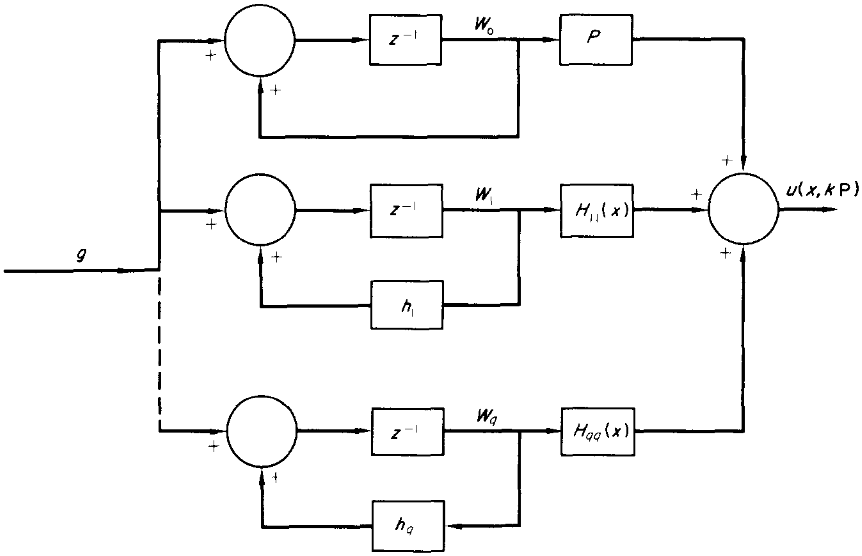

FIG. 1. State variable diagram for $u(x, kP)$.

It is seen from equations (20) that at equlibrium, the state variable W_0 is the only nonzero state since at equilibrium g must equal zero. Hence any nonzero initial condition as given by equation (2a) may now be imposed on the state equations as given by equations (20) and (21). Hence it follows from equation (21) that

$$W_0(0) = u_i/P \tag{22}$$

and also from equations (20) that

$$W_j(0) = 0, \quad j = 1, 2, \ldots, q. \tag{23}$$

Hence starting with any initial condition for the temperature the solution proceeds in a straightforward manner using a digital computer. For the parameters as considered by Crosbie and Viskanta [1], the results were in excellent agreement. For most cases the sampling period was taken to be 0·0010 and 7 terms were retained in the infinite series. The results were obtained directly and hence no iteration techniques were needed. The accuracy is dependent upon the sampling period chosen and the number of terms retained in the infinite series. In Table 1 is shown the results for 7 terms retained and $P = 0{\cdot}0010$ along with the results obtained for 5 terms retained and $P = 0{\cdot}0020$. There is very little benefit in retaining for this particular case more than 5 terms in the series. The method of trial and error is perhaps the simplest to employ since the sampling period may be readily changed by merely changing one card in the program: also extra terms may be readily added from the infinite series as the equations for each term follow a distinct pattern. For certain cases where the dimensionless time required to heat or cool the slab is long, then for these cases the sampling period may be increased considerably as the input function $g(kP)$ is relatively slow changing, and hence one need sample

Table 1. Comparison of solutions using the sampled-data model with those of Crosbie and Viskanta [1] for $N_{rh} = 0{\cdot}4$, $N_{Bi} = 0{\cdot}5$ and $\theta_h = 0{\cdot}2$

Time (t)	Surface temperature $u(1, t)$ Crosbie and Viskanta	Case A*	Case B*
0·00	0·2000	0·2000	0·2000
0·25	0·5563	0·5564	0·5566
0·50	0·6670	0·6671	0·6672
0·75	0·7488	0·7489	0·7490
1·00	0·8115	0·8116	0·8117
1·25	0·8592	0·8593	0·8594
1·50	0·8951	0·8951	0·8952
1·75	0·9219	0·9219	0·9220
2·00	0·9419	0·9419	0·9420

Case A* is for $P = 0{\cdot}0010$ with 7 terms retained while Case B* is for $P = 0{\cdot}0020$ with 5 terms retained in the infinite series.

less frequently than for cases where the function $g(kP)$ is changing faster with respect to time. Also the effects of time varying fluid and environmental temperatures as well as having the radiation exchange factor and the heat-transfer coefficient explicit functions of the surface temperature may be simply incorporated into the system model equations by merely changing the $g(kP)$ expression. The remainder of the program remains the same.

CONTINUOUS-TIME MODEL EQUATIONS

The discrete-time equations as given by equations (20) and (21) may be readily converted to a system of continuous-time equations for analog computer study. Knowing the sampling period P the conversion process leads to the following set of equations

$$\frac{\mathrm{d}W_i(t)}{\mathrm{d}t} = \lambda_i W_i(t) + B_i g(t) \tag{24}$$

where

$$\lambda_i = \frac{(h_i - 1{\cdot}0)}{P}; \qquad i = 1, 2, \ldots, q \tag{25}$$

$$B_i = 1{\cdot}0/P = B \quad \text{for all } i \tag{26}$$

and

$$\lambda_0 = 0{\cdot}0. \tag{27}$$

Hence the matrix form of these equations is given by

$$\dot{\boldsymbol{W}}(t) = \boldsymbol{\Phi W}(t) + \boldsymbol{B}g(t) \tag{28}$$

where $\boldsymbol{\Phi}$ is the diagonal matrix representing the eigenvalues of the distributed system and $\boldsymbol{B}$ is a constant column vector. The equation for the temperature at any point x is

$$u(x, t) = P\,W_0(t) + H_{11}(x)\,W_1(t) + \ldots + H_{qq}(x)\,W_q(t). \tag{29}$$

After time and magnitude scaling the equations were programmed on an analog computer and gave essentially the same results as for the discrete-time equations. It was not observed but it would be expected that the analog model equations would give slightly better results as we have a continuous monitoring of the input signal $g(t)$. However, since the sampling period P was chosen so small (0·0010) little or no difference was detected.

CONCLUDING REMARKS

It has been shown how a representative sampled-data system of equations representing a distributed parameter system lends itself to readily obtainable solutions using either a digital or analog computer. The state equations in the variables $\boldsymbol{W}$ are seen to represent each eigenvalue retained in the infinite series, and also the term t which appears explicitly in the solution of equation (1) with the given boundary conditions. Hence each state equation retained represents the contribution of the particular eigenvalue to the overall temperature response of the system. Results obtained were in excellent agreement with published results.

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